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EXISTENCE, UNIQUENESS AND APPROXIMATION FOR GENERALIZED SADDLE POINT PROBLEMS

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ABSTRACT

Extensions of the well known results of Brezzi on saddle point problems are presented. The class of problems is generalized to include the unsymmetric case, and the known stability and approximation results are strengthened, and applied to the generalized problem. As an application, an existence theorem for the Stokes problem is given.

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1. Introduction

In [4], F. Brezzi obtained existence, uniqueness and stability results for the saddle point problem

$$a(u,v) + b(\varphi,v) = \langle f,v\rangle \qquad \forall\ v \in \mathcal{V} \tag{1.1}$$

$$b(\psi,u) = \langle g,\psi\rangle \qquad \forall\ \psi \in \mathcal{S} \tag{1.2}$$

where (1.1) - (1.2) are to be solved for the pair $(u,\varphi) \in \mathcal{V} \times \mathcal{S}$ and where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bilinear forms defined respectively on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{S} \times \mathcal{V}$. $\langle f,\cdot\rangle$ and $\langle g,\cdot\rangle$ denote bounded linear functionals respectively in the dual spaces $\mathcal{V}'$ and $\mathcal{S}'$ of the Hilbert spaces $\mathcal{V}$ and $\mathcal{S}$. In addition, approximation of (1.1) - (1.2) in closed subspaces $\mathcal{V}^h \subset \mathcal{V}$ and $\mathcal{S}^h \subset \mathcal{S}$ was considered. In essence, the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ were assumed to satisfy the usual type of continuity and stability conditions (see §2 below), and the stability and error estimates provided were then obtained in the norm on $\mathcal{V} \times \mathcal{S}$

$$\|(u,\varphi)\| = \|u\|_{\mathcal{V}} + \|\varphi\|_{\mathcal{S}}. \tag{1.3}$$

Further questions studied in [4] concern "numerical integration" and "nonconforming" approximations, both relevant to finite element choices of the approximating subspaces $\mathcal{V}^h$ and $\mathcal{S}^h$.

It is well known that many problems of interest arise naturally in the form (1.1) - (1.2). Among them are the governing equations of linear elasticity appropriately expressed, and the Stokes problem for slow viscous incompressible flow of a Newtonian fluid. Also, many variational problems with equality constraints give rise to Euler equations of this type once a Lagrange multiplier is introduced. Another source of problems of this type is in the reduction of higher order boundary value problems to lower order ones by the introduction of new variables representing derivatives of the original variables ("mixed methods"). Some of these topics are considered in [4] and the references therein. See also [5], [6] and especially their references for later work.

There are two objections one may raise concerning the theory of [4]. The first of them concerns the use of the product space norm (1.3). This is clearly the natural norm to use if the view is taken of (1.1) - (1.2) that (under appropriate conditions) its left side defines an operator

$$M : \mathcal{U} \times \mathcal{S} \to \mathcal{U} \times \mathcal{S},$$

that is, that the nature of the coupling between the variables u and φ is, in a sense, to be suppressed. In many physical examples this is not a satisfactory approach, and it would seem worthwhile to have a theory in which the stability and approximation estimates were expressed separately for the variables u, and φ. Second, some problems of interest [e.g. 7] have the more general form than (1.1) - (1.2),

$$a(u,v_2) + b_1(\varphi,v_2) = \langle f,v_2\rangle \qquad \forall\ v_2 \in \mathcal{V}_2 \tag{1.4}$$

$$b_2(\psi_2,u) = \langle g,\psi_2\rangle \qquad \forall\ \psi_2 \in \mathcal{S}_2 \tag{1.5}$$

$u \in \mathcal{V}_1$, $\varphi \in \mathcal{S}_1$ being the sought solution, $a(\cdot,\cdot)$, $b_1(\cdot,\cdot)$, $b_2(\cdot,\cdot)$ being defined on the implied Hilbert spaces and satisfying certain continuity and stability conditions. The results of [4] do not, in general, apply to this problem. The object of the present note is, therefore, to obtain existence uniqueness and approximation results for the system (1.4) - (1.5) taking regard, where possible, of the separate identities of the u and φ variables. The desired estimates and general approximation scheme are obtained and presented in §2 - §4. In the final section, as an application of the theory, a proof of an existence theorem for the Stokes problem mentioned above is given. In a forthcoming report [8], the results are applied to a wider set of problems, including questions of approximation not considered here.

2. Preliminaries

Standard notations are used throughout. All Hilbert spaces are assumed real. For such a space $\mathcal{U}$ its inner product and norm will be denoted by $(\ ,\ )_{\mathcal{U}}$ and $\|\ \|_{\mathcal{U}}$ respectively. The subscript will be omitted when the context makes it possible.

$\mathcal{U}'$ denotes the dual space of $\mathcal{U}$ and the value of $f \in \mathcal{U}'$ at $u \in U$ will be written as $\langle f,u \rangle_{\mathcal{U}'\mathcal{U}}$. The norm of $f \in \mathcal{U}'$ is written as $\|f\|_{\mathcal{U}'}$. Again, subscripts will be omitted where possible. By definition,

$$\|f\|_{\mathcal{U}'} = \sup_{\substack{u \in \mathcal{U} \\ u \neq 0}} \frac{\langle f,u \rangle}{\|u\|} \qquad f \in \mathcal{U}'.$$

The main objective of this section is to prove a verison of Babuska's generalization of the Lax-Milgram theorem, appropriate to the problem we wish to solve. The following elementary lemma will be needed.

Lemma 2.1

Let $c(\theta,v)$ be a bilinear form defined $\forall\ \theta \in \mathcal{S}$ and $v \in \mathcal{V}$, $\mathcal{S}$ and $\mathcal{V}$ being Hilbert spaces. Let

$$Z = \{z \in \mathcal{V} | c(\theta,z) = 0 \quad \forall\ \theta \in \mathcal{S}\}$$

and suppose $c(\theta,v)$ is continuous in the sense

$$|c(\theta,v)| \leq K\|\theta\|\ \|v\| \quad \forall\ \theta \in \mathcal{S},\ v \in \mathcal{U}. \tag{2.1}$$

Then $\mathcal{V} = Z \oplus \mathcal{W}$ where $\mathcal{W} \equiv Z^{\perp}$.

Proof:

It is necessary only to show that Z is a closed linear subspace of $\mathcal{V}$, and this follows immediately from the continuity of $c(\theta,v)$.

The required form of Babuska's theorem is the following.

Theorem 2.1

Let $c(\theta,v)$ satisfy, in addition to the conditions of Lemma 2.1,

$$\sup_{\|\theta\|=1} c(\theta,w) \geq \gamma\|w\|_{\mathcal{V}} \quad \forall\, w \in \mathcal{W}; \qquad \sup_{w\in\mathcal{W}} c(\theta,w) > 0, \ \forall\, \theta \in \mathcal{S} \ \ \theta\neq 0 \qquad (2.2)$$

with $\gamma > 0$.
Then the equation for $w \in \mathcal{W}$

$$c(\theta,w) = \langle f,\theta\rangle \qquad \forall\, \theta \in \mathcal{S}$$

is uniquely solvable $\forall\, f \in \mathcal{S}'$ and the stability estimate

$$\|w\|_{\mathcal{V}} \leq \gamma^{-1}\|f\|$$

holds.

Proof.

$\mathcal{W}$ is a Hilbert space, with the norm and inner product inherited from $\mathcal{V}$. The bilinear form

$$\tilde{c}(\theta,w) \triangleq c(\theta,w) \qquad \forall\, \theta \in \mathcal{S}, \quad w \in \mathcal{W}$$

satisfies the conditions of Babuska's theorem [1] $\forall\, \theta \in \mathcal{S}$ and $w \in \mathcal{W}$. The desired result therefore follows from [1].

Corollary 2.1

The equation for $\theta \in \mathcal{S}$

$$c(\theta,w) = \langle g,w\rangle \qquad \forall\, w \in \mathcal{W}, \ \ g \in \mathcal{V}'$$

is uniquely solvable $\forall\ g \in \mathcal{V}'$ and a constant γ' exists such that

$$\|w\|_{\mathcal{V}} \leq (\gamma')^{-1}\|g\|.$$

Proof.

The theorem implies the existence of a constant, γ', such that

$$\sup_{\|w\|_{\mathcal{V}}} c(\theta,w) \geq \gamma'\|\theta\| \quad \forall\ \theta \in \mathcal{S} \quad \gamma' > 0,$$

and the result follows from the theorem by applying the latter to the form $\bar{c}(w,\theta) \equiv c(\theta,w)$, observing that $g \in \mathcal{V}' \Rightarrow g \in \mathcal{W}'$ with $\|g\|_{\mathcal{W}'} \leq \|g\|_{\mathcal{V}'}$.

If Z is empty, Theorem 2.1 reduces to Babuska's theorem.

3. Existence, Uniqueness, Stability

Let $b_1(\theta_1,v_2)$ and $b_2(\theta_2,v_1)$ be defined as indicated on Hilbert spaces $\mathcal{S}_1,\mathcal{S}_2,\mathcal{V}_1,\mathcal{V}_2$ and satisfy the conditions of Theorem 2.1 with constants K_i,γ_i,γ_i', $i = 1,2$. Let Z_i, $i = 1,2$ denote the associated closed null spaces and let $\mathcal{W}_i = Z_i^{\perp}$, $i = 1,2$. Let $a(v_1,v_2)$ be another bilinear form defined $\forall\ v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$ and satisfying the conditions of boundedness and stability

$$|a(v_1,v_2)| \leq M\|v_1\|\ \|v_2\| \quad \forall\ v_i \in \mathcal{V}_i \quad i = 1,2 \tag{3.1}$$

$$\sup_{\|z_2\|=1} a(z_1,z_2) \geq \delta\|z_1\|_{\mathcal{V}_1} \quad : \quad \sup_{z_1} a(z_1,z_2) > 0 \tag{3.2}$$

where (3.2) hold respectively $\forall\ z_1 \in Z_1$ and $z_2 \in Z_2$, $z_2 \neq 0$, $\delta > 0$.

Consider the (generalized) saddle point problem, of finding $u \in \mathcal{V}_1$, $\varphi \in \mathcal{S}_1$ such that

$$a(u,v_2) + b_1(\varphi,v_2) = \langle f,v_2\rangle \quad \forall\ v_2 \in \mathcal{V}_2 \tag{3.3}$$

$$b_2(\psi_2,u) = \langle g,\psi_2\rangle \quad \forall\ \psi_2 \in \mathcal{S}_2 \tag{3.4}$$

where $f \in \mathcal{V}_2'$ and $g \in \mathcal{S}_2'$.

Theorem 3.1

Under the hypotheses on $b_i(\theta_i,v_i)$ and $a(v_1,v_2)$ stated above, the saddle problem (3.3) - (3.4) has a unique solution $u \in \mathcal{V}_1$, $\varphi \in \mathcal{S}_1$, and there exist constants c_{ij}, $i,j = 1,2$ such that the stability estimates

$$\|u\| \leq c_{11}\|f\| + c_{12}\|g\| \qquad \|\varphi\| \leq c_{21}\|f\| + c_{22}\|g\|.$$

hold, where $c_{11} = \delta^{-1}, c_{12} = \gamma_2^{-1}(M+1)$, $c_{21} = (\gamma_1')^{-1}(1+Mc_{11}), c_{22} = (\gamma_1')^{-1}Mc_{12}$.

Proof.

Z_i are Hilbert spaces with inner products inherited from $\mathcal{V}_i$, $i = 1,2$. By (3.1)

$$a(z_1,z_2) \leq M\|z_1\|\ \|z_2\| \tag{3.5}$$

and hence, by (3.5), (3.2) and Corollary 2.1 with $\mathcal{S} = Z_1$, $\mathcal{V} = Z_2$, $c(\cdot,\cdot) = a(\cdot,\cdot)$ and Z empty it follows that the equation

$$a(\zeta, z_2) = \langle h, z_2 \rangle, \; h \in Z_2', \qquad \forall \; z_2 \in Z_2 \tag{3.6}$$

has a unique solution ζ satisfying

$$\|\zeta\|_{Z_1} \equiv \|\zeta\|_{\mathcal{V}_1} \leq \delta^{-1} \|h\|_{Z_2'} \tag{3.7}$$

and in particular, if $h \in \mathcal{V}_2'$ then $h \in Z_2'$, $\|h\|_{Z_2'} \leq \|h\|_{\mathcal{V}_2'}$ and

$$\|\zeta\|_{\mathcal{V}_1} \leq \delta^{-1} \|h\|_{\mathcal{V}_2'}.$$

Let w be the unique solution to (3.4) in $\mathcal{W}_1$ whose existence and boundedness follows from Theorem 2.1. By (3.1)

$$\begin{aligned} a(w, v_2) &\leq M\|w\| \; \|v_2\| \qquad \forall \; v_2 \in \mathcal{V}_2 \\ &\leq M\gamma_2^{-1}\|g\| \; \|v_2\| \end{aligned} \tag{3.8}$$

so that $a(w,\cdot)$ defines a bounded linear functional on $\mathcal{V}_2$ and hence on Z_2. Therefore,

$$\langle f, \cdot \rangle - a(w, \cdot) \tag{3.9}$$

defines a bounded linear functional on Z_2. Hence, by (3.6), the equation

$$a(\zeta, z_2) = \langle f, z_2 \rangle - a(w, z_2) \qquad \forall \; z_2 \in Z_2 \tag{3.10}$$

has a unique solution $\zeta \in Z_1$ bounded as at (3.7). In fact

$$\|\zeta\| \leq \delta^{-1} [\|f\| + M\gamma_2^{-1} \|g\|].$$

Define $u = w+\zeta \in \mathcal{V}_1$. Clearly,

$$\|u\| \leq \gamma_2^{-1}\|g\| + \delta^{-1}[\|f\|+M\gamma_2^{-1}\|g\|] \tag{3.11}$$

so that $a(u,\cdot)$ defines a bounded linear functional on $\mathcal{V}_2$. By Corollary 2.1, with $c(\cdot,\cdot) = b_1(\cdot,\cdot)$ $\mathcal{V} = \mathcal{V}_2$, $\mathcal{S} = \mathcal{S}_1$, it follows that the equation

$$b_1(\varphi,w_2) = \langle f,w_2\rangle - a(u,w_2) \tag{3.12}$$

has a unique solution $\varphi\in\mathcal{S}_1$ satisfying

$$\|\varphi\| \leq (\gamma_1')^{-1}[\|f\|+M\|u\|]. \tag{3.13}$$

Transposing the $a(\cdot,\cdot)$ terms on the right of (3.10) and (3.12) to the left sides and adding gives

$$a(u,w_2+z_2) + b_1(\varphi,w_2+z_2) = \langle f,w_2+z_2\rangle.$$

Since $\mathcal{V}_2 = \mathcal{W}_2 \oplus Z_2$ this is equivalent to

$$a(u,v_2) + b_1(\varphi,v_2) = \langle f,v_2\rangle \quad \forall\ v_2\in\mathcal{V}_2 \tag{3.14}$$

and since, clearly,

$$b_2(\psi_2,w+\zeta) = b_2(\psi_2,w) = \langle g,\psi_2\rangle \quad \forall\ \psi_2\in\mathcal{S}_2, \tag{3.15}$$

(3.14) - (3.15) show the existence of the solution $u\in\mathcal{V}_1$, $\varphi\in\mathcal{S}_1$. Uniqueness is clear from the above proof since w is unique but in any case can be simply derived in the usual way from the stability estimates (3.11) and (3.13) which may be conveniently rewritten in the form

$$\|u\| \le c_{11}\|f\| + c_{12}\|g\|, \quad \|\varphi\| \le c_{21}\|f\| + c_{22}\|g\|.$$

Thus, the theorem is proved. Notice that e.g. the stability of u is independent of the properties of $b_1(\psi,v)$. This and similar facts would be lost in a product norm type of analysis.

It is worth observing that if $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$ and $a(\cdot,\cdot)$ is known to be continuous on $\mathcal{V} \times \mathcal{V}$ and to satisfy

$$a(v,v) \ge \delta\|v\|^2, \quad \delta > 0 \qquad \forall\ v \in \mathcal{V}, \tag{3.16}$$

then conditions (3.2) will hold.

4. Approximation

Let $\mathcal{V}_i^h$, $\mathcal{S}_i^h$, $i = 1,2$ be closed subspaces of $\mathcal{V}_i$, $\mathcal{S}_i$, $i = 1,2$ respectively. The approximate problem to be considered is to find $u^h \in \mathcal{V}_1^h$ and $\varphi^h \in \mathcal{S}_1^h$ such that

$$a(u^h,v_2^h) + b_1(\varphi^h,v_2^h) = \langle f,v_2^h\rangle \qquad \forall\ v_2^h \in \mathcal{V}_2^h \tag{4.1}$$

$$b_2(\psi_2^h,u^h) = \langle g,\psi_2^h\rangle \qquad \forall\ \psi_2^h \in \mathcal{V}_2^h \tag{4.2}$$

(4.1) - (4.2) constitute a generalized saddle problem on the indicated Hilbert spaces, $\mathcal{V}_i^h$, $\mathcal{S}_i^h$, $i = 1,2$. In order to show the existence, uniqueness and stability of a solution to this problem, we simply require that the forms $a(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$ satisfy on their respective spaces the conditions which enabled the corresponding "continuous" problem to be solved in Theorem 3.1. In fact, the continuity conditions are automatically satisfied, since $\mathcal{V}_i^h \in \mathcal{V}_i$ and $\mathcal{S}_i^h \in \mathcal{S}_i$, $i = 1,2$. We have then the orthogonal decompositions

$$\mathcal{V}_i^h = Z^h \oplus \mathcal{W}^h \qquad i = 1,2$$

where

$$Z_1^h = \{ z_1^h \in \mathcal{V}_1^h \mid b_2(\psi_2^h, z_1^h) = 0 \quad \forall \ \psi_2^h \in \mathcal{S}_2^h \}$$

with a corresponding definition for Z_2^h. In general, $Z_i^h \not\subset Z_i$, $\mathcal{W}_i^h \not\subset \mathcal{W}_i$, $i = 1,2$.

The following conditions on the forms and spaces are now assumed to hold: with $\delta_h > 0$

$$\sup_{\|z_2^h\|=1} a(z_1^h, z_2^h) \geq \delta_h \|z_1^h\| \ : \quad \sup_{z_1^h} a(z_1^h, z_2^h) > 0 \tag{4.3}$$

with $z_1^h \in Z_1^h$, $z_2^h \in Z_2^h$, $z_2^h \neq 0$. These are analogous to (3.2). Also the conditions, with $\gamma_1^h > 0$ and $\gamma_2^h > 0$

$$\sup_{\|\theta_1^h\|=1} b_1(\theta_1^h, w_2^h) \geq \gamma_1^h \|w_2^h\| \ : \quad \sup_{w_2^h} b_1(\theta_1^h, w_2^h) > 0 \tag{4.4}$$

$\forall \ w_2^h \in \mathcal{V}_2^h$ and $\theta_1^h \neq 0 \in \mathcal{S}_1^h$, respectively, and

$$\sup_{\|\theta_2^h\|=1} b_2(\theta_2^h, w_1^h) \geq \gamma_2^h \|w_1^h\| \ : \quad \sup_{w_1^h} b_2(\theta_2^h, w_1^h) > 0 \tag{4.5}$$

$\forall \ w_1^h \in \mathcal{V}_1^h$ and $\theta_2^h \neq 0 \in \mathcal{S}_2^h$ respectively are assumed to hold. In this case, an appeal to Theorem 3.1 establishes the existence uniqueness and stability of a solution to (4.1) - (4.2). In the particular case in which $\mathcal{V}_1 = \mathcal{V}_2$, $\mathcal{V}_1^h = \mathcal{V}_2^h$ the condition (4.3) will be a consequence of the coercivity condition (3.16), if the latter holds.

Next, we shall estimate the differences $\|u-u^h\|_{\mathcal{V}_1}$ and $\|\varphi-\varphi^h\|_{\mathcal{S}_1}$.

<u>Theorem 4.1</u>

There exist numbers L_i, $i = 1,2,3,4$ dependent only on $(\gamma_1^h)'$, γ_2^h, K_2, K_1, δ_h, M (and defined below) such that

$$\|u-u^h\| \leq L_1 \inf_{\hat{u}^h \in \mathcal{V}_1^h} \|u-\hat{u}^h\| + L_2 \inf_{\varphi^h \in \mathcal{S}_1^h} \|\varphi-\hat{\varphi}^h\|$$

$$\|\varphi-\varphi^h\| \leq L_3 \inf_{\hat{\varphi}^h \in \mathcal{S}_1^h} \|\varphi-\hat{\varphi}^h\| + L_4 \inf_{\hat{u}^h \in \mathcal{V}_1^h} \|u-\hat{u}^h\|.$$

<u>Proof</u>

$u = \zeta + w$, where $\zeta \in Z_1$, $w \in \mathcal{W}_1$. Analogously, u^h has the unique decomposition $u^h = \zeta^h + w^h$, with $\zeta^h \in Z_1^h$ and $w^h \in \mathcal{W}_1^h$. Moreover, by subtraction of (4.2) and (3.4)

$$b_2(\psi_2^h, w-w^h) = 0 \quad \forall\ \psi_2^h \in \mathcal{S}_2^h. \tag{4.6}$$

Let $\hat{u}^h$ be arbitrary in $\mathcal{V}_1^h$, $\hat{u}^h = \hat{z}^h + \hat{w}^h$, with $\hat{z}^h \in Z_1^h$, $\hat{w}^h \in \mathcal{W}_1^h$. Then by (4.6)

$$b_2(\psi_2^h, \hat{w}^h - w^h) = b_2(\psi_2^h, \hat{u}^h - u)$$

and by (4.5) and continuity,

$$\gamma_2^h \|\hat{w}^h - w^h\| < K_2 \|u - \hat{u}^h\|. \tag{4.7}$$

Subtracting (4.1) and (3.3) and taking $v_2^h \epsilon Z_2^h$, it follows that

$$a(u-u^h, v_2^h) = -b_1(\varphi - \hat{\varphi}^h, v_2^h)$$

where $\hat{\varphi}^h$ is arbitrary in S_1^h. Therefore,

$$a(\hat{u}^h - u^h, v_2^h) = a(\hat{u}^h - u, v^h) - b_1(\varphi - \hat{\varphi}^h, v_2^h)$$

so that

$$a(\hat{z}^h - \zeta^h, v_2^h) = a(\hat{u}^h - u, v_2^h) + a(w^h - \hat{w}^h, v_2^h) - b_1(\varphi - \hat{\varphi}^h, v_2^h).$$

By (4.3), (3.1) and continuity of $b_1(\cdot,\cdot)$

$$\delta_h \|\zeta^h - \hat{z}^h\| \leq M(\|\hat{u}^h - u\| + \|w^h - \hat{w}^h\|) + K_1 \|\varphi - \hat{\varphi}^h\|. \tag{4.8}$$

Then, since

$$\|u-u^h\| \leq \|u - \hat{u}^h\| + \|\hat{u}^h - u^h\| \leq \|u - \hat{u}^h\| + \|\hat{z}^h - \zeta^h\| + \|\hat{w}^h - w^h\|$$

(4.7) - (4.8) give

$$\|u-u^h\| \leq L_1 \|u - \hat{u}^h\| + L_2 \|\varphi - \hat{\varphi}^h\|$$

where

$$L_1 = 1 + K_2/\gamma_2^h + M/\delta_h + (MK_2)/(\gamma_2^h \delta_h), \quad L_2 = K_1/\delta_h.$$

The first part of the theorem now follows, since $\hat{u}^h$ and $\hat{\varphi}^h$ are arbitrary.

To prove the second part, again from (4.1) and (3.3), with v_2^h now arbitrary in $\mathcal{V}_2^h$,

$$b_1(\varphi-\varphi^h, v_2^h) = -a(u-u^h, v_2^h)$$

so that $\forall\ \hat{\varphi}^h \in \mathcal{S}_1^h$,

$$b_1(\hat{\varphi}^h-\varphi^h, v_2^h) = b_1(\hat{\varphi}^h-\varphi, v_2^h) - a(u-u^h, v_2^h).$$

Then by (4.4) and continuity

$$(\gamma_1^h)' \|\hat{\varphi}^h-\varphi^h\| \leq K_1\|\hat{\varphi}^h-\varphi\| + M\|u-u^h\|,$$

and by the triangle inequality and the first part of the theorem,

$$\|\varphi-\varphi^h\| \leq L_3\|\varphi-\hat{\varphi}^h\| + L_4\|u-\hat{u}^h\|$$

where $L_3 = 1 + K_1/(\gamma_1^h)' + ML_2/(\gamma_1^h)'$, $L_4 = ML_1/(\gamma_1^h)'$. Thus, Theorem 4.1 is proved.

Corollary 4.1

If $Z^h \subset Z$, then the estimate for $\|u-u^h\|$ may be improved to

$$\|u-u^h\| \leq L_1 \inf_{\hat{u}^h \in \mathcal{V}_1^h} \|u-\hat{u}^h\|.$$

Proof

This follows from (4.8), since the term $K_1\|\varphi-\hat{\varphi}^h\|$ no longer needs to appear in this inequality, and the subsequent deductions from it.

Sometimes, the norms in which the problem is well posed (i.e. the norms on $\mathcal{V}_i$ and $\mathcal{S}_i$, $i = 1,2$) are not those in which approximation errors are wanted. Several approaches to this question are possible and have been recently investigated, [2], [5], [6].

5. Stokes' Problem

As a simple application of Theorem 3.1, consider the Stokes problem of finding u and p such that

$$-\nu\Delta u + \nabla p = f \quad \text{in } \Omega \tag{5.1}$$

$$\operatorname{div} u = 0 \tag{5.2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{5.3}$$

Here, Ω is a bounded domain of R^3, and ν a positive constant.

In order to formulate the weak problem, the spaces $\mathcal{V}$ and $\mathcal{S}$ must be defined. $\mathcal{V}$ will be defined as a certain closed linear subspace of $\vec{H}_0^1(\Omega)$, where $\vec{H}_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ functions in the norm

$$\|u\|^2 = \int_\Omega \nabla u \cdot \nabla u \; \partial\Omega. \tag{5.4}$$

(Derivatives are always assumed to be taken in the distribution sense). Let

$$Z_0^\infty = \{z \in \vec{C}_0^\infty(\Omega) \mid \text{div } z = 0\}$$

$$\mathfrak{w}_0^\infty = \{w \in \vec{C}_0^\infty(\Omega) \mid \text{curl } w = 0\},$$

and denote by $\tilde{Z}$ and $\tilde{\mathfrak{w}}$ the completion of Z_0^∞ and $\mathfrak{w}_0^\infty$ in the norm (5.4). $\tilde{Z}$ and $\tilde{\mathfrak{w}}$ are then closed linear subspaces of $\vec{H}_0^1(\Omega)$.

Observe that $\forall\ z \in \tilde{Z}, w \in \tilde{\mathfrak{w}}$

$$\text{div } z = 0, \quad \text{curl } w = 0; \tag{5.5}$$

for if, e.g., $\{w_m\} \to w$ in $\vec{H}_0^1(\Omega)$, with $w_m \in C_0^\infty(\Omega)$, then

$$\|\text{curl } w\|_{\vec{L}^2(\Omega)} = \|\text{curl}(w-w_m)\|_{\vec{L}^2(\Omega)} \leq K\|w-w_m\|_{\vec{H}_0^1(\Omega)}$$

from which the second equation follows. The first may be shown by a similar argument.

Lemma 5.1

$\forall\ u,v \in \vec{H}_0^1(\Omega)$

$$(\nabla u, \nabla v)_{\vec{L}^2(\Omega)} = (\text{div } u, \text{div } v)_{L^2(\Omega)} + (\text{curl } u, \text{curl } v)_{\vec{L}^2(\Omega)} \tag{5.6}$$

Proof

It is sufficient to prove this result for $u,v \in C_0^\infty(\Omega)$. To do this, take the common identity

$$\text{curl curl } v = \nabla \text{ div } v - \text{div } \nabla v,$$

multiply on the left by u and integrate over Ω using the boundary conditions on u and v and the easily provable identity

$$\int_\Omega v\cdot \text{curl } u \, \partial\Omega = \int_\Omega (\text{curl } v)\cdot u \, \partial\Omega + \int_{\partial\Omega} (v\times u)\cdot n \partial s$$

which is valid on $\vec{H}_0^1(\Omega)$, and the tensor identity

$$\int_\Omega v\cdot \text{div } T \, \partial\Omega = -\int_\Omega \nabla v\cdot T \, \partial\Omega + \int_{\partial\Omega} (v\cdot T) n \partial s$$

valid for second order tensors T with components in $H_0^1(\Omega)$. The result is (5.6).

<u>Lemma 5.2</u>

$\tilde{Z}$ and $\tilde{\mathfrak{W}}$ are orthogonal in $\vec{H}_0^1(\Omega)$

<u>Proof</u>

Recall that $\vec{H}_0^1(\Omega)$ is a Hilbert space with inner product

$$(u,v) = \int_\Omega \nabla u\cdot \nabla v \partial\Omega, \quad u,v\in\vec{H}_0^1(\Omega). \tag{5.7}$$

Then if $z\in\tilde{Z}$ and $w\in\tilde{\mathfrak{W}}$ the result follows from (5.5) - (5.6). The space $\mathfrak{V}_1 = \mathfrak{V}_2 = \mathfrak{V}$, which will be denoted by $\vec{\mathfrak{H}}_0^1(\Omega)$, is defined as the following closed linear subspace of $\vec{H}_0^1(\Omega)$,

$$\vec{\mathfrak{H}}_0^1(\Omega) \triangleq \tilde{Z} \oplus \tilde{\mathfrak{W}}, \tag{5.8}$$

so that by definition, $u\in\vec{\mathfrak{H}}_0^1(\Omega) \Leftrightarrow$

$$u = z + w \quad z\in\tilde{Z},\ w\in\tilde{\mathfrak{W}}.$$

This decomposition is unique, since $\tilde{\mathcal{Z}}$ and $\tilde{\mathfrak{W}}$ are closed subspaces of the Hilbert space $\vec{\mathcal{H}}^1_0(\Omega)$ consisting of (5.8) with the inner product (5.7).

<u>Lemma 5.3</u>

Let $u \in \vec{\mathcal{H}}^1_0(\Omega)$. Then if $\operatorname{div} u = 0$, then $u \in \tilde{\mathcal{Z}}$: if $\operatorname{curl} u = 0$, then $u \in \tilde{\mathfrak{W}}$.

<u>Proof</u>

Let $u = z + w$. Then $\operatorname{div} u = 0 \Rightarrow \operatorname{div} w = 0$. However, from (5.6), with $u = v = w$ it then follows that

$$\|w\|^2_{\vec{H}^1_0(\Omega)} = \|\operatorname{div} w\|^2_{L^2(\Omega)} = 0$$

so that $w = 0$, and $u = z \in \tilde{\mathcal{Z}}$. The other result may be proved similarly.

The following characterizations of $\tilde{\mathcal{Z}}$ and $\tilde{\mathfrak{W}}$ may therefore be given: $u(\in \vec{\mathcal{H}}^1_0(\Omega)) \in \tilde{\mathcal{Z}} \Leftrightarrow \operatorname{div} u = 0$ and $u \in \tilde{\mathfrak{W}} \Leftrightarrow \operatorname{curl} w = 0$.

For the scalar space, $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2$, we take $\mathcal{L}^2(\Omega)$ defined as

$$\mathcal{L}^2(\Omega) = \{\varphi \in L^2(\Omega) \mid \int_\Omega \varphi \partial\Omega = 0\}$$

with inner product and norm inherited from $L^2(\Omega)$. Since $\mathcal{L}^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, it is a Hilbert space.

With the spaces $\mathcal{V} \equiv \vec{\mathcal{H}}^1_0(\Omega)$ and $\mathcal{S} \equiv \mathcal{L}^2(\Omega)$ defined, we turn now to verification of the various hypotheses. The weak form of the problem will be the following: find $u \in \vec{\mathcal{H}}_0^1(\Omega)$ and $\varphi \in \mathcal{L}^2(\Omega)$ such that

$$
\begin{aligned}
a(u,v) + b(\varphi,v) &= \langle f,v \rangle \qquad && \forall\ v \in \vec{\mathcal{H}}_1^0(\Omega) \\
b(\psi,u) &= 0 && \forall\ \psi \in \mathcal{L}^2(\Omega)
\end{aligned}
$$

where the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as follows:

$$
\begin{aligned}
a(u,v) &= \nu \int_\Omega \nabla u \cdot \nabla v \ \partial\Omega \qquad && \forall\ u,v \in \vec{\mathcal{H}}_0^1(\Omega) \\
b(\psi,v) &= -\int_\Omega \psi \ \mathrm{div}\ v \ \partial\Omega && \forall\ \psi \in \mathcal{L}^2(\Omega),\ v \in \vec{\mathcal{H}}_0^1(\Omega).
\end{aligned}
\tag{5.9}
$$

First of all, from

$$
\begin{aligned}
|b(\psi,v)| &\leq \|\psi\|_{L^2(\Omega)} \|\mathrm{div}\ v\|_{L^2(\Omega)} \\
&\leq \|\psi\|_{\mathcal{L}^2(\Omega)} \|v\|_{\mathcal{H}_0^1(\Omega)}
\end{aligned}
$$

the continuity of $b(\psi,v)$ follows. The spaces Z and $\mathfrak{W}$ are therefore defined. In fact, from the definition of $b(\cdot,\cdot)$ in (5.9) and Lemma 5.3, it is clear that $Z \equiv \tilde{Z}$. Moreover, since $\mathfrak{W}$ is defined as $Z^\perp$ in $\vec{\mathcal{H}}_0^1(\Omega)$, $\mathfrak{W} \equiv \tilde{\mathfrak{W}}$. Thus, Z and $\mathfrak{W}$ are known. Since $w \in \vec{H}_0^1(\Omega)$,

$$
\int_\Omega \mathrm{div}\ w \ \partial\Omega = \int_\Omega w \cdot n \ \partial s = 0,
$$

and it follows that $\mathrm{div}\ w \in \mathcal{L}^2(\Omega)\ \forall\ w \in \mathfrak{W}$. But then

$$\sup_{\|\theta\|=1} \int_\Omega \theta \text{ div } w \, \partial\Omega = \|\text{div } w\|_{L^2(\Omega)}$$

and by the identity (5.6).

$$\|\text{div } w\|_{L^2(\Omega)} = \|w\|_{\vec{H}_0^1(\Omega)}$$

so that the condition (2.2) is proved for $b(\cdot,\cdot)$. To prove (2.3), assume that it is not true. Then $\theta_0 \in \mathcal{L}^2(\Omega)$ exists such that

$$\int_\Omega \theta_0 \text{div } w \, \partial\Omega = 0 \qquad \forall \; w \in \mathfrak{b}. \tag{5.10}$$

Now the equation

$$\text{div } u = h, \; h \in C_0^\infty(\Omega), \; \int_\Omega h \, \partial\Omega = 0$$

is always solvable for $u \in C_0^\infty(\Omega)$, since we can always solve

$$\text{div}(\nabla\varphi) = h, \; \varphi \in C_0^\infty(\Omega)$$

and choose $u = \nabla\varphi$. But then $u \in \vec{C}_0^\infty(\Omega)$, and curl $u = 0$. Hence, $u \in \mathfrak{b}$. Then by (5.10), θ_0 is orthogonal to $\{h \in C_0^\infty(\Omega) \mid \int_\Omega h \partial\Omega = 0\}$ which is a set dense in $\mathcal{L}^2(\Omega)$. Hence $\theta_0 = 0$, and it follows that (2.3) is satisfied. It remains to verify the conditions on $a(\cdot,\cdot)$. But these are immediate since $\upsilon > 0$. Thus, all the conditions of Theorem 3.1 are verified and the existence, uniqueness and stability of a solution to the weak problem follows.

Investigation of numerical schemes will not be carried out here. However, it must be mentioned that the discrete stability conditions (4.4) - (4.5) by no means follow from the continuous ones. For example, it is easy to see that the only piecewise linear solenoidal field vanishing on the boundary of a square triangulated into smaller squares, each bisected by a rightward sloping diagonal in the plane, is the zero field. Then Z^h contains only the zero field and the discrete version of (2.2) can clearly never be true. Thus, each choice of spaces $\mathcal{V}^h$, $\mathcal{S}^h$ requires independent verification. Some choices for $\mathcal{V}^h$, $\mathcal{S}^h$ which are apparently suitable for use are given in [3]. These topics are considered more fully in [8].

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